BASIC KNOWLEDGE for Integer Solutions (Part 1)

(1) Positive integer solutions

Let n and r be positive integers. The number of positive integer solutions to the equation

$$x_1 + x_2 + x_3 + \cdots + x_r = n$$

is
$$N = \binom{n-1}{r-1} \qquad (1)$$

(2) Non-negative integer solutions

Let n and r be positive integers. The number of non-negative integer solutions to the equation

$$x_1 + x_2 + x_3 + \cdots + x_r = n$$

is
$$N = \binom{n+r-1}{r-1} \qquad (2)$$

(The following formulas (3) to (10) were derived by me. I have not seen them explicitly stated in another math book).

(3) Positive integer solutions with restrictions

Let n and r be positive integers. $1 \le x_i \le 6$. The number of positive integer solutions to the equation

$$x_1 + x_2 + x_3 + \cdots + x_r = n$$

is
$$N = \binom{n-1}{r-1} - \binom{r}{1}\binom{n-1\times 6-1}{r-1} + \binom{r}{2}\binom{n-2\times 6-1}{r-1} - \binom{r}{3}\binom{n-3\times 6-1}{r-1} + \ldots \qquad (3)$$

(4) The number of ways to roll a sum of *n* with three dice

The number of solutions to the equation $x_1 + x_2 + x_3 = n$ (with restrictions $1 \le x_i \le 6$)

is
$$N = \binom{n-1}{3-1} - \binom{3}{1}\binom{n-1\times 6-1}{3-1} + \binom{3}{2}\binom{n-2\times 6-1}{3-1} \qquad (4)$$

(5) The number of ways to roll a sum of *n* with four dice

Number of solutions to the equation $x_1 + x_2 + x_3 + x_4 = n$ (with restrictions $1 \le x_i \le 6$)

is $N = \binom{n-1}{4-1} - \binom{4}{1}\binom{n-1\times 6-1}{4-1} + \binom{4}{2}\binom{n-2\times 6-1}{4-1} - \binom{4}{3}\binom{n-3\times 6-1}{4-1}$ (5)

(6) The number of ways to roll a sum of *n* with five dice

Number of solutions to the equation $x_1 + x_2 + x_3 + x_4 + x_5 = n$ (with the restrictions $1 \le x_i \le 6$) is

$$N = \binom{n-1}{5-1} - \binom{5}{1}\binom{n-1\times 6-1}{5-1} + \binom{5}{2}\binom{n-2\times 6-1}{5-1} - \binom{5}{3}\binom{n-3\times 6-1}{5-1} + \binom{5}{4}\binom{n-4\times 6-1}{5-1} \quad (6)$$

(7) The number of 3-digit positive integers with the sum of the digits *n*

Number of solutions to the equation: $x_1 + x_2 + x_3 = n$ (with the restrictions $1 \le x_1 \le 9$, $0 \le x_2 \le 9$, and $0 \le x_3 \le 9$):

$$N = N_1 + N_2 \quad (7)$$

We calculate N_1, the number of positive integer solutions by using the formula

$$N_1 = \binom{n-1}{2} - \binom{3}{1}\binom{n-1\times 9-1}{2} + \binom{3}{2}\binom{n-2\times 9-1}{2}$$

N_2 is the number of solutions containing zero(s) in the ten's and unit digits that can be obtained by listing.

(8) The number of *r*-digit positive integers with the sum of the digits *n*

The number of solutions to the equation: $x_1 + x_2 + x_3 + \ldots + x_r = n$ (with the restrictions $1 \le x_1 \le 9$, and $0 \le x_i \le 9$, $i = 2, 3, \ldots, r$):

$$N = N_1 - N_2 \quad (8)$$

N_1, the number of non-negative integer solutions of the above equation (with the restrictions $0 \le x_i \le 9$, $i = 1, 2, 3, \ldots, r$):

$$N_1 = \binom{n+r-1}{r-1} - \binom{r}{1}\binom{n+r-1\times 10-1}{2} + \binom{r}{2}\binom{n+r-2\times 10-1}{2} - \ldots \quad (9)$$

N_2, the number of non-negative integer solutions to the equation: $y_1 + y_2 + \ldots + y_s = n$ (with the restrictions $0 \le y_i \le 9$, $i = 1, 2, 3, \ldots, s$ and $s = r - 1$):

$$N_2 = \binom{n+s-1}{s-1} - \binom{s}{1}\binom{n+s-1\times 10-1}{s-1} + \binom{s}{2}\binom{n+s-2\times 10-1}{s-1} - \ldots \quad (10)$$

PROBLEMS

Problem 1. Three friends have a total of 6 identical pencils, and each one has at least one pencil. In how many ways can this happen? (2004 AMC 8).

Problem 2. Three fair, standard six-faced dice of different colors are rolled. In how many ways can the dice be rolled such that the sum of the numbers rolled is 10? (2003 Mathcounts States).

Problem 3. Three fair, standard six-faced dice of different colors are rolled. In how many ways can the dice be rolled such that the sum of the numbers rolled is 15?

Problem 4. Three fair, standard six-faced dice of different colors are rolled. In how many ways can the dice be rolled such that the sum of the numbers rolled is 8?

Problem 5. Four fair, standard six-faced dice of different colors are rolled. In how many ways can the dice be rolled such that the sum of the numbers rolled is 20?

Problem 6. How many 3-digit numbers are there such that the sum of the digits is 11?

Problem 7. How many 3-digit numbers are there such that the sum of the digits is 17?

Problem 8. How many 3-digit numbers are there such that the sum of the digits is 6?

Problem 9. There are 10 apples to be given to 7 boys. Each boy needs to get at least one apple. How many ways to distribute the apples?

Problem 10: Find the number of nonnegative integer solutions to

$$2x_1 + x_2 + x_3 + x_4 + x_5 + x_6 + x_7 + x_8 + x_9 + x_{10} = 3$$

Problem 11. 20 identical computers are to be distributed into 5 distinct rooms so each room receives at least 2 computers. Find the number of such distributions.

Problem 12. When five standard six-sided dice are rolled sequentially there are $6^5 =$ 7776 possible outcomes. For how many outcomes is the sum of the five rolled numbers exactly 27? (2007 Mathcounts State Sprint #22).

Problem 13. How many positive three-digit numbers can be formed such that the sum of the three digits is a prime number less than 10? (1997 Mathcounts State).

Problem 14. How many four-digit integers greater than 5000 are there for which the thousands digit equals the sum of the other three digits? (1996 Mathcounts National #30).

SOLUTIONS

Problem 1. Solution: **10.**

Since each person must have at least one pencil, no one has zero pencils. In other words, this problem is seeking for the number of positive integer solutions to the equation $x_1 + x_2 + x_3 = 6$. Using the formula in section 1, our solution is

$$\binom{n-1}{r-1} = \binom{6-1}{3-1} = \binom{5}{2} = 10.$$

Problem 2. Solution: **27.**

This is a problem seeking the number of positive integer solutions to the equation

$x_1 + x_2 + x_3 = 10$

with restrictions $1 \le x_i \le 6$

Using the formula 4 in section 1, the number of solutions is

$$N = \binom{10-1}{2} - 3\binom{10-6-1}{2} = 36 - 9 = 27.$$

Problem 3. Solution: **10.**

$x_1 + x_2 + x_3 = 15$ with $1 \le x_i \le 6$

Using the formula 4 in section 1, the number of solutions is

$$\text{N} = \binom{15-1}{3-1} - \binom{3}{1}\binom{15-6-1}{3-1} + \binom{3}{2}\binom{15-6-6-1}{3-1} = 10.$$

Another way to think about this question:

The original question can become "How many nonnegative integer solutions are there to the equation $x_1 + x_2 + x_3 = 3$?"

Using the formula in section 1, we have $\binom{5}{2} = 10$.

Problem 4. Solution: **21.**

$x_1 + x_2 + x_3 = 8 \qquad 1 \le x_i \le 6$

We don't need to use the formula 4 in section 1 because we can just find the number of positive integer solutions because we have $1 \le x_i \le 6$. This means that if $x_1 = 1, x_2 = 1$ then x_3=6, which does not exceed the given range for x_i

$N = \binom{8-1}{3-1} = \binom{7}{2}$.

Problem 5. Solution: **35.**

Method 1:

$x_1 + x_2 + x_3 + x_4 = 20 \;\; 1 \le x_i \le 6$

$$\text{N} = \binom{20-1}{4-1} - \binom{4}{1}\binom{20-6-1}{4\text{-}1} + \binom{4}{2}\binom{20-2\times 6-1}{4-1} = 35.$$

Method 2: This problem is the same as "How many nonnegative solutions are there to the equation $x_1 + x_2 + x_3 + x_4 = 4$?"

$$N = \binom{4+4-1}{4-1} = \binom{7}{3} = 35.$$

Problem 6. Solution: **61.**

Method 1:

To find the number of solutions to $x_1 + x_2 + x_3 = 11$ with restrictions $1 \le x_1 \le 9$, $0 \le x_2 \le 9$, and $0 \le x_3 \le 9$, we first calculate the number of positive integer solutions:

$$N_1 = \binom{11-1}{2} = \binom{10}{2} = 45.$$

We still need to, however, calculate N_2, the number of 3-digit numbers containing zero in the ten's or one's digits. This can easily be done by listing. We know that ten's and one's digit can't both be zero since they must add up to 11. The 3-digit numbers containing zero in the units digit can be listed below:

9	2	0	(4 rearrangements: 920, 902, 290, 209)
8	3	0	(4 rearrangements)
7	4	0	(4 rearrangements)
6	5	0	(4 rearrangements)

So $N_2 = 16$.

$N = N_1 + N_2 = 45+16 = 61$.

Method 2:

The number of solutions to the equation: $x_1 + x_2 + x_3 = 11$ (with the restrictions $1 \le x_1 \le 9$, and $0 \le x_i \le 9$, $i = 2, 3$):

$$N = N_1 - N_2$$

N_1, the number of non-negative integer solutions of the above equation:

$$N_1 = \binom{11+3-1}{3-1} - \binom{3}{1}\binom{11+3-1\times 10-1}{3-1} = 78 - 9 = 69$$

N_2, the number of non-negative integer solutions to the equation: $y_1 + y_2 = 11$ (with the restrictions $0 \le y_i \le 9$, $i = 1, 2$):

$$N_2 = \binom{11+2-1}{2-1} - \binom{2}{1}\binom{11+2-1\times 10-1}{2-1} = 12 - 4 = 8$$

The desired solution is then $N_1 - N_2 = 69 - 8 = 61$.

Problem 7. **Solution**: **61.**

Method 1:

To find the number of solutions to $x_1 + x_2 + x_3 = 17$ with restrictions $1 \le x_1 \le 9$, $0 \le x_2 \le 9$, and $0 \le x_3 \le 9$, we first calculate the number of positive integer solutions:

$$N_1 = \binom{17-1}{2} - \binom{3}{1}\binom{17-9-1}{2} = \binom{16}{2} - 3\binom{7}{2} = 120 - 3\times 21 = 57 .$$

We then calculate N_2, the number of positive integers containing zero in the ten's and units digits by listing:

9	8	0	(4 rearrangements)

So $N_2 = 4$

$N = N_1 + N_2 = 57 + 4 = 61$.

Method 2:

The number of solutions to the equation: $x_1 + x_2 + x_3 = 17$ (with the restrictions $1 \le x_1 \le 9$, and $0 \le x_i \le 9$, i = 2, 3):

$$N = N_1 - N_2$$

N_1, the number of non-negative integer solutions of the above equation:

$$N_1 = \binom{17+3-1}{3-1} - \binom{3}{1}\binom{17+3-1\times 10-1}{3-1} = 171 - 108 = 63$$

N_2, the number of non-negative integer solutions to the equation: $y_1 + y_2 = 17$ (with the restrictions $0 \le y_i \le 9$, i = 1, 2):

$$N_2 = \binom{17+2-1}{2-1} - \binom{2}{1}\binom{17+2-1\times 10-1}{2-1} = 18 - 16 = 2$$

The desired solution is then $N_1 - N_2 = 63 - 2 = 61$.

Problem 8. **Solution**: **<u>21.</u>**

To find the number of solutions to $x_1 + x_2 + x_3 = 6$ with restrictions $1 \le x_1 \le 9$, $0 \le x_2 \le 9$, and $0 \le x_3 \le 9$, we first calculate the number of positive integer solutions:

$$N_1 = \binom{6-1}{2} = \binom{5}{2} = 10 .$$

Then we calculate N_2, the number of positive integers containing zero in the ten's and units digits by listing:

6	0	0	(1 rearrangement)
5	1	0	(4 rearrangements)
4	2	0	(4 rearrangements)
3	3	0	(2 rearrangements)

So $N_2 = 11$

$N = N_1 + N_2 = 10 + 11 = 21$.

Problem 9. Solution: **84.**

Method 1: We can write an equation such as $x_1 + x_2 + x_3 + x_4 + x_5 + x_6 + x_7 = 10$

Our job is to find the number of positive solutions of above equation:

$$N = \binom{n-1}{r-1} = \binom{10-1}{7-1} = \binom{9}{6} = 84.$$

Method 2: If we give one apple to each boy first, then there will be 3 apples left for next distribution. These 3 apples can be given to 3 boys, 2 boys, or one boy. Then our job is to find the nonnegative solutions to the equation $x_1 + x_2 + x_3 + x_4 + x_5 + x_6 + x_7 = 3$, which is $\binom{9}{6} = 84$.

Problem 10. **Solution**: **174.**

If $x_1 = 0$, then $x_2 + x_3 + x_4 + x_5 + x_6 + x_7 + x_8 + x_9 + x_{10} = 3$

The number of nonnegative solutions is $\binom{11}{3} = 165$

If $x_1 = 1$, then $x_2 + x_3 + x_4 + x_5 + x_6 + x_7 + x_8 + x_9 + x_{10} = 1$

The number of nonnegative solutions is $\binom{9}{1} = 9$

The total number of solutions is 165 + 9 = 174.

Problem 11. Solution: **1001.**

Method 1: We have $x_1 + x_2 + x_3 + x_4 + x_5 = 20$ with the restriction $x_i \ge 2$.
We let $x_i = y_i + 1$, then we have $y_1 + y_2 + y_3 + y_4 + y_5 = 15$ with restriction: $y_i \ge 1$. The number of positive integer solutions is: $\binom{15-1}{5-1} = \binom{14}{4} = 1001$.

Method 2: We distribute 2 computers to each room first, and then the question becomes to find the number of nonnegative solution to the equation:

$a+b+c+d+e=10$

The desired solution is $\binom{5+10-1}{10}=\binom{14}{4}=1001$.

Problem 12. Solution: **35.**

Method 1: The number of solutions to $x_1+x_2+x_3+x_4+x_5=27$ with restrictions $1 \le x_i \le 6$ is

$$N=\binom{27-1}{4}-\binom{5}{1}\binom{27-6-1}{4}+\binom{5}{2}\binom{27-6-6-1}{4}-\binom{5}{3}\binom{27-6-6-6-1}{4}=35.$$

Method 2: Another way to think about this question is, "How many nonnegative integer solutions are there to the equation $x_1+x_2+x_3+x_4+x_5=3$?"

The answer is $\binom{3+5-1}{5-1}=\binom{7}{4}=35$.

Problem 13. Solution: 52.

We have:

$$x_1+x_2+x_3=7 \quad (1)$$
$$x_1+x_2+x_3=5 \quad (2)$$
$$x_1+x_2+x_3=3 \quad (3)$$
$$x_1+x_2+x_3=2 \quad (4)$$

For equation (1), we have $N_1=\binom{7-1}{3-1}=\binom{6}{2}=15$;

N_2:	7	0	0	(1 rearrangement)
	6	1	0	(4 rearrangements)
	5	2	0	(4 rearrangements)
	4	3	0	(4 rearrangements)

For equation (2), we have $N_1 = \binom{5-1}{3-1} = \binom{4}{2} = 6$;

N_2:	5	0	0	(1 rearrangement)
	4	1	0	(4 rearrangements)
	3	2	0	(4 rearrangements)

For equation (3), we have $N_1 = \binom{3-1}{3-1} = \binom{2}{2} = 1$;

N_2:	3	0	0	(1 rearrangement)
	2	1	0	(4 rearrangements)

For equation (4), we have $N_1 = \binom{2-1}{3-1} = 0$;

N_2:	2	0	0	(1 rearrangement)
	1	1	0	(2 rearrangements)

The answer is the sum of everything: (15 +13) + (6 + 9) + (1 + 5) + 3 = 52.

Problem 14. Solution: <u>185.</u>

This problem is the same as to find the non-negative solutions of

$x + y + z = 5 \quad \Rightarrow \binom{7}{2}$

$x + y + z = 6 \quad \Rightarrow \binom{8}{2}$

$x + y + z = 7 \quad \Rightarrow \binom{9}{2}$

$x + y + z = 8 \quad \Rightarrow \binom{10}{2}$

$x + y + z = 9 \quad \Rightarrow \binom{11}{2}$

The solution is the sum of everything: $\binom{7}{2} + \binom{8}{2} + \binom{9}{2} + \binom{10}{2} + \binom{11}{2} = 185$.